Lecture 25: Distributions Teaser

- Distributions further generalize the concept of weak solutions.

Recall that we "tested" $f \in L^1_{loc}$ by looking at $\int f \varphi \, dx$, $\varphi \in C_c^\infty$.

We may then interpret f as a map $C_c^\infty \to \mathbb{C}$ given by

$$\varphi \mapsto \int f \varphi \, dx$$

- A distribution is a more general linear functional $C_c^\infty \to \mathbb{C}$. The term "distribution" was inspired by charge distributions in electrostatics:

Model Problem: Coulomb's Law

- Coulomb's law of electrostatics says that a particle with electric charge q_0, located at the origin, generates an electric field

$$E(x) = \frac{k q_0 x}{|x|^3} \quad \text{for a constant } k.$$

- In the previous lecture, we discussed Gauss' law

$$\nabla \cdot E = 4\pi k \rho \quad \text{for } \rho \text{ the charge per unit}$$

volume.

- Coulomb's field isn't differentiable at 0, but away from 0,

$$\nabla \cdot \frac{x}{r^3} = \frac{\nabla \cdot x}{r^3} - \frac{3x}{r^4} \nabla r = 3/r^3 - \frac{3x}{r^4} \cdot \frac{x}{r} = 0,$$

Consistent with Gauss' prediction in that charge density at a point is 0.

Hence, Coulomb predicts charge only at 0, but then as a function in L^1_{loc}, this is 0. To reconcile this, consider the weak form of Gauss' law

$$\int_{\mathbb{R}^3} E \cdot \nabla \varphi \, dx = -4\pi k \int_{\mathbb{R}^3} \rho \varphi \, dx$$

for all $\varphi \in C_c^\infty$.

• Since E is smooth except at 0, we integrate-by-parts away from 0.

$$\int_{\mathbb{R}^3} E\cdot\nabla\psi\,dx = \lim_{\varepsilon\to 0}\int_{\{r\geq\varepsilon\}} E\cdot\nabla\psi\,dx$$

$$= \lim_{\varepsilon\to 0}\int_{\{r=\varepsilon\}} \eta\cdot E\psi\,dS \quad - 0$$

$$= \lim_{\varepsilon\to 0}\int_{\{r=\varepsilon\}} -\frac{kq_0}{\varepsilon^2}\psi\,dS \quad (\eta = -x/r)$$

Since ψ is continuous, this limit approaches $-kq_0(4\pi)\psi(0)$ (b/c $\frac{1}{4\pi\varepsilon^2}\int_{\{r=\varepsilon\}}\psi\,dS \to \psi(0)$)

The weak condition then requires

$$\int_{\mathbb{R}^3}\rho\psi\,dx = q_0\psi(0)$$

for every $\psi\in C_c^\infty$, or that ρ is a charge at the origin of magnitude q_0!

This "point density" is often called the Dirac Delta function $\delta(x)$ defined so

$$\int_{\mathbb{R}^n} f(x)\,\delta^{(n)}(x)\,dx = f(0)$$

for $f\in C^0$.

However, δ is not a function and this isn't an integral! We consider δ as the map $f\mapsto f(0)$.

Roughly speaking, the above gives $\nabla\cdot\frac{x}{r^3} = 4\pi\delta$

The Space of Distributions.

- A distribution on a domain $U \subseteq \mathbb{R}^n$ is a continuous linear functional $C_c^\infty(U) \to \mathbb{C}$. We usually write its evaluation as a pairing

$$\varphi \mapsto \langle u, \varphi \rangle \quad \text{for } \varphi \in C_c^\infty.$$

Linearity means

$$\langle u, c_1\varphi_1 + c_2\varphi_2 \rangle = c_1 \langle u, \varphi_1 \rangle + c_2 \langle u, \varphi_2 \rangle$$

for all $c_1, c_2 \in \mathbb{C}$, $\varphi_1, \varphi_2 \in C_c^\infty$.

Continuity means that if $\varphi_k \to \varphi$, then $\langle u, \varphi_k \rangle \to \langle u, \varphi \rangle$.
But what convergence do we want for $\varphi_k \to \varphi$?
This convergence in C_c^∞ means that for some $K \subset U$ compact, $\mathrm{supp}(\varphi_j), \mathrm{supp}(\varphi) \subseteq K$ and
$\varphi_j \to \varphi$, $\partial^\alpha \varphi_j \to \partial^\alpha \varphi$ uniformly on K for all α.

- The set of distributions forms a vector space that we denote $\mathcal{D}'(U)$. The theory of this space was developed independently by Sergei Sobolev & Laurent Schwartz.

- functions in L^1_{loc} tie to distributions as we saw before: for $f \in L^1_{loc}$
$$\langle f, \varphi \rangle = \int_U f\varphi \, dx$$

- Convergence in the space $\mathcal{D}'(U)$ is defined weakly: $u_k \to u$ if $\langle u_k, \varphi \rangle \to \langle u, \varphi \rangle$ for all $\varphi \in C_c^\infty$

$\boxed{\text{Lemma}}$ Given $f \in L^1(\mathbb{R}^n)$ satisfying $\int_{\mathbb{R}^n} f\,dx = 1$, define
$$f_\varepsilon(x) = \varepsilon^n f(\varepsilon x), \quad \text{for } \varepsilon > 0.$$

Then, $\lim_{a \to \infty} f_a = \delta$.

$\boxed{\text{Pf}}$ For $\psi \in C_c^\infty$

$$\langle f_a, \psi \rangle = \int_{\mathbb{R}^n} a^n f(ax)\, \psi(x)\, dx$$
$$= \int_{\mathbb{R}^n} f(x)\, \psi(x/a)\, dx$$

so $\langle f_a, \psi \rangle - \psi(0) = \int_{\mathbb{R}^n} f(x) \left[\psi(x/a) - \psi(0) \right] dx$

~~Since ψ is compactly supported, $\psi(x/a) > 0$ only for $x/a \in B$~~ Pick $\varepsilon > 0$.

Pick $R > 0$ so $\int_{\mathbb{R}^n \setminus B(0,R)} f(x)\, dx < \varepsilon$.

Then, pick $\gamma > 0$ so if $|y| < \gamma$,
$|\psi(y) - \psi(0)| < \varepsilon$.

we have that for large a, $|x/a| \leq R/a < \gamma$

and so

$$|\langle f_a, \psi \rangle - \psi(0)| \leq \int_{B(0,R)} |f(x)|\varepsilon\, dx + \int_{\mathbb{R}^n \setminus B(0,R)} |f(x)| \cdot 2\|\psi\|_\infty\, dx$$
$$\leq \varepsilon + 2\varepsilon \|\psi\|_\infty \qquad \square$$

So $\lim_{a \to \infty} \langle f_a, \psi \rangle = \psi(0)$.

e.g.) $H_t(x) = \frac{1}{(\sqrt{4\pi t})^n} e^{-|x|^2/4t}$

is a rescaling of $\frac{1}{(\sqrt{4\pi})^n} e^{-|x|^2/4}$

and $\lim_{t \to 0} H_t = \delta$

- Distributional Derivatives are defined such that for $u \in \mathcal{D}'(U)$,

$$\langle D^\alpha u, \varphi \rangle = (-1)^{|\alpha|} \langle u, D^\alpha \varphi \rangle$$

(like for weak derivatives)

•) For example, we previously saw that

for $\mathcal{H}(x) = \begin{cases} 1 & x > 0 \\ 0 & x < 0 \end{cases}$

$\mathcal{H}'(x) = -\delta_0(x)$ in this sense:

$\langle \mathcal{H}, \varphi' \rangle = \int_0^\infty \varphi'(x)dx = -\varphi(0) = \langle -\delta, \varphi \rangle$

Fundamental Solutions

- Consider L a differential operator, such as Δ.

Assume we can solve for Φ so

$$L\Phi = \delta.$$

Then, if we consider any PDE $\{ Lu = f$, we have that (within some assumptions) $u = \Phi * f$ has

$$Lu = L\left[\int \Phi(x-y)f(y)dy\right] = \int L\Phi(x-y)f(y)dy$$

$$= \int \delta(x-y)f(y)dy = f(x)$$

- Transitioning to incorporating boundary data may take many forms. For Laplace's Equation, one uses Green's Functions.

- For evolution equations, one often uses the concept of a forward-in-time fundamental solution such as $E_+(t,x) = \chi_{[0,\infty)}(t) \cdot H_t(x)$ for the heat equation. Then,

$$u(t,x) = f *_{t,x} E_+ + g *_x E_+(t,\cdot)$$

solves $\begin{cases} (\partial_t - \Delta)u = f \\ u(0,x) = g(x) \end{cases}$

- Adding this unit jump provides the general version of Duhamel's Principle.